LINEAR ALGEBRA MID TERM EXAM

This exam is of **30 marks** and is **3 hours long** - from 10 am to 1pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly.

1. Let V_N be the vector space of polynomials over the real numbers of degree $\leq N$ over $\mathbb{R}$. Let a and b in $\mathbb{R}$ and define a linear functional on V_N by

$$\lambda(p) = \int_a^b p(t)dt$$

Let $D : V_N \to V_N$ be the differentiation operator on V_N.

- What is the matrix of D with respect to the usual basis of V_N? (3)
- What is $\det(D)$ (2)
- Is D invertible? (1)
- What is $D^*(\lambda)$, where D^* is the transpose. (4)

2. Let σ be a permutation of degree n, that is, an element of the symmetric group $\mathfrak{S}_n$. If A is an $n \times n$ matrix with rows $\alpha_1, \dots, \alpha_n$, let $\sigma(A)$ be the matrix with rows $\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(n)}$.

- Show that $\sigma(AB) = \sigma(A)B$. In particular $\sigma(A) = \sigma(I)A$. (4)
- Is $\sigma^{-1}(I)$ the inverse of $\sigma(I)$? Justify your answer. (3)
- Is $\sigma(A)$ similar to A? Justify your answer. (3)

3. Let V_N be the polynomials over $\mathbb{C}$ of degree $\leq N$. Let $x_0, x_1, \dots, x_n$ be in $\mathbb{C}$. Consider the linear map

$$\Phi : V_N \to \mathbb{C}^{n+1}$$

$$\phi(P) = (P(x_0), \dots, P(x_n))$$

- Compute the matrix of Φ with respect to the standard bases of V_N and $\mathbb{C}^{n+1}$. (3)
- Show that the elements $1, (x - x_0), (x - x_0)(x - x_1), \dots, (x - x_0)(x - x_1) \cdots (x - x_{n-1})$ form a basis for V_N. (3)
- Compute the change of basis matrix and used that to compute the determinant of Φ. (4)